A Finite Group Analysis of the Quark Mass Matrices and Flavor Dynamics

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Abstract

We perform a finite group analysis on the quark mass matrices. We argue that the dominant terms should be proportional to class operators of the group and that symmetry breaking to split the mass spectrum and simultaneous diagonalizability to suppress flavor changing neutral currents can be accomplished at this point. The natural setting is a multi-scalar model and the scalar doublets can have masses of the weak scale without any parameter tuning. When we specialize to S_3 as the group of choice, we arrive at the results that the dominant mass terms are ‹democratic› and that the ratios of light masses and the Cabbibo angle $\cong (\frac{m_d}{m_s})^{\frac{1}{2}}$ are all given by group parameters in the breaking of S_3 to S_2. A large mass expansion is then performed and a generalized Wolfenstein parameterization is given. Further breaking by way of introducing heavy-light transitions in the down-type mass matrix is here related to the heavy-light Cabbibo-Kobayashi-Maskawa elements.

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One of the frontiers in understanding elementary interactions is the organization of fermion masses, which in some effective way are related to Yukawa couplings between fermions and scalars. Many proposals have been made and most are motivated by some conjectures on physics at a much higher energy scale. Typically, a certain 'texture' is assumed for the Yukawa structure and then a renormalization group analysis is performed to predict consequences for physical processes which are currently experimentally reachable. These are very ambitious and formidable endeavours.

We shall take a different tack in the present discussion. Our starting point is to accept what we know from the data about fermion masses and mixing between up and down sectors at the electroweak scale. Several features stand out: the almost decoupling of the top and bottom heavy quarks from the lighter ones, the high degree of suppression of flavor changing neutral currents at low energies, and the validity of the Wolfenstein parameterization. We then ask the question: How much of this can be understood by applying symmetry considerations? We argue in this note that one can achieve quite a lot in this regard. Of course, some assumptions need to be made along the way, and they will be explicitly stated. They have to do with symmetry breaking, which should be familiar to most of us, drawing upon past experience. We remark that this approach may be complementary to the top down method just mentioned. One advantage here is an immediate link between physical parameters and those introduced in the group analysis.

Before being specific, let us outline how such an analysis is developed. Consider a group with a finite number of elements g_i. We can partition these elements into disjoint conjugate classes C_j. Because C_j commute with each other and can be made hermitian, they are a part of the complete set of observables and can be used to label states.[1] Also, because all elements of the group commute with these class operators, C_j's are invariants. As a zeroth order approximation, i.e., before symmetry breaking is introduced, the interaction which is responsible for mass generation for either charged $\frac{2}{3}$ or $-\frac{1}{3}$ type

quarks is a linear combination of these class operators, which we write generically as

$$M_0 = \sum a_j C_j. \tag{1}$$

Because we are dealing with a finite group, the elements g_i can be made unitary, and the invariance under the proposed symmetry is

$$g_i M_0 g_i^{-1} = M_0. \tag{2}$$

The spectrum of M_0, which splits quarks into heavy and light species, generally has some degeneracy at this level. Past experience leads us to speculate that the degeneracy is lifted by symmetry breaking along some directon in the group space. Thus, one assumes that another term

$$M_1 = \sum b_k g_k, \tag{3}$$

accounts for that, where the sum is over a set of elements, such that symmetry of some subgroup remains. Therefore, M_1 must be expressible as a function of the class operators of the subgroup. This forces conditions on b's, reducing their independent number.

We must digress at this point to discuss the problem of flavor changing neutral currents. As one follows the discussion so far, one must wonder about the mechanisms which cause the division of M into M_0 and M_1. The current lore is that there may be different $SU(2)$ Higgs doublets, which couple separately to M_0 and M_1. We accept this and will not be discussing the dynamical details pertaining to such scalars at this juncture. The only issue we want to bring up is that if the scalars are distinct, they will generally introduce tree level flavor changing neutral current processes.[2] The reason is that if we write out the scalars explicitly, we have

$$M(x) = \sum a_j' C_j \phi_0(x) + \sum b_k' g_k \phi_1(x), \tag{4}$$

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where the first and second terms on the right hand side, respectively, come from M_0 and M_1. Fermion masses are induced by replacing the fields with their vacuum expectation values

$$\phi_{0,1} \to v_{0,1}, \tag{5}$$

and performing a bi-unitary transformation $U^\dagger M V$. Because of the space-time dependence, such a transformation cannot diagonalize $M(x)M(x)^\dagger$ for all x, unless

$$M_0 M_1, M_0 M_1^\dagger, M_1 M_1^\dagger, \tag{6}$$

commute. We recall that $M_0 = M_0^\dagger$, and $[M_0, M_1] = 0$ is automatic by the very nature of M_0 being made of class operators. Commutativity would be trivial if $M_1 = M_1^\dagger$ also. However, in order to lift all degeneracies at this point, hermiticity of M_1 may not be warranted and commutativity should be checked. If satisfied, then under rather general scalar self interaction, the dominant part of the induced flavor changing neutral currents can in fact be avoided at least up to the one loop level.[3] We call the commutativity requirement radiatively natural. The gist is due to a result that the otherwise worrisome divergent pieces of the one loop contributions can be absorbed into wave function renormalizations without spoiling simultaneous diagonalizability.

We have generated masses for the heavy quarks through M_0, and masses for the light quarks and their mixing mostly through M_1. The requirement of simultaneous diagonalizability probably will not induce misalignment between the heavy and the light states of the up and down type quarks if we assume that the symmetry basis vectors in both sectors are the same; i.e., the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements $V_{td,ts,cb,ub}$ vanish at this level. If our picture is in concordance with nature, there must exist another piece M_2, which gives rise to finite, albeit small, heavy-light mixing matrix elements, and which also results in flavor violation in heavy-light transitions. We shall now turn to an example to give some specifics.

A finite group which is suggested empirically is the symmetric group S_3,[4] with group elements $\{e, (12), (13), (23), (123), (132)\}$, where e is the identity, (12) is the operation of exchanging entries in positions 1 and 2, and (123) corresponds to $1 \to 2 \to 3 \to 1$, etc. Let us take the up quark sector 3x3 mass matrix

$$\bar{u}_L M_u u_R, \tag{7}$$

which we assume to be invariant under

$$\bar{u}_L \to \bar{u}_L g_i, \quad u_R \to g_i^{-1} u_R, \tag{8}$$

for $g_i \epsilon S_3$. The conjugate classes are $\{e\}, \{(12), (13), (23)\}$, and $\{(123), (132)\}$, with the concomitant class operators

$$C_1 = e, \quad C_2 = (12) + (13) + (23), \quad C_3 = (123) + (132). \tag{9}$$

From the group table, one finds $C_3 = (C_2)^2/3 - C_1$, which means that at most two of these class operators need be specified to label states.

The three quark states are assumed to be linear combinations of the basis vectors $|\alpha, \alpha, \beta>$, $|\alpha, \beta, \alpha>$, and $|\beta, \alpha, \alpha>$, on which the symmetry operations act on the entries α and β, e. g.

$$(13)(|\alpha, \alpha, \beta>, \ |\alpha, \beta, \alpha>, |\beta, \alpha, \alpha>)$$
$$= (|\beta, \alpha, \alpha>, |\alpha, \beta, \alpha>, |\alpha, \alpha, \beta>)$$
$$= (|\alpha, \alpha, \beta>, \ |\alpha, \beta, \alpha>, |\beta, \alpha, \alpha>) \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{10}$$

from which one obtains the (reducible) matrix representation. One can easily show that on these states, the class operator

$$C_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{11}$$

and $C_1 + C_3 = C_2$. Looking at their eigenvalues, one sees that C_2 has (0,0,3), which makes it empirically rather compelling to take[4]

$$M_0 = m_0 C_2, \tag{12}$$

to give mass to the top quark, where m_0 is a real constant carrying the dimension of mass.

To account for the light quarks c and u, we *assume* that M_1 is along some direction such that S_2 is the residual symmetry. For S_2, there are only two elements $\{e, g\}$, with $g^2 = e$. To make this general, we write

$$M_1 = m_1 g, \quad g = a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132), \tag{13}$$

where $m_1 \ll m_0$ is another real constant with the dimension of mass. A set of conditions which yield the requirement $g^2 = e$ is

$$a_1 = 0, \quad a_5 + a_6 = 0, \quad a_2 + a_3 + a_4 = 1,$$

and

$$a_2^2 + a_3^2 + a_4^2 = 1 + 2a_6^2 \tag{14}$$

We shall make the choice that all the a's are real. (This results in a non-hermitian reducible g, which is what we need to separate the light masses. The residual symmetry acts on the mass matrix $M_0 + M_1$, but not on the states.) It is easy to verify that the simultaneous diagonalizability conditions of Eq.(6) are satisfied, basically because M_0 is unitarily equivalent to a diagonal matrix with only one non-vanishing entry. The eigenvalues of $M_1 M_1^\dagger$ are

$$\lambda_{1,2}^2 = m_1^2(1 + 6a_6^2 \mp 2a_6\sqrt{3 + 9a_6^2}), \quad \Delta\lambda_3^2 = m_1^2, \tag{15}$$

which depend on a_6 only. One can solve for it as

$$a_6 = \frac{m_c - m_u}{2\sqrt{3m_u m_c}}. \tag{16}$$

The corresponding eigenvectors are

$$|\lambda_{1,2}^0> = N_{1,2}|x_{1,2}, \; y_{1,2}, \; -(x_{1,2}+y_{1,2})>, \quad |\lambda_3^0> = \frac{1}{\sqrt{3}}|1, \; 1, \; 1>,$$

with

$$\frac{y_{1,2}}{x_{1,2}} = \frac{\mp\sqrt{3+9a_6^2}+3a_4-1}{3a_2-1}, \tag{17}$$

and $N_{1,2}$ are normalization factors.

With the conditions of Eq.(14) and the a's being real, we have three independent parameters, which may be chosen as m_1, a_2 and a_6. They uniquely give the masses $m_u \cong \lambda_1$, $m_c \cong \lambda_2$ and the relative weight y/x of the physical states $|\lambda_{1,2}^0> \cong |u,c>$. We can replicate the same analysis for the down sector and obtain similar results, which we use primes to denote. A further assumption of charge independence $a_2 = a_2'$ reduces the number of parameters to five, which is in agreement with the count of $m_{c,u}, m_{s,d}$ and the Cabibbo angle $sin\theta_c \cong V_{us} \cong < \lambda_1^0|\lambda_2'^0 > .$

A particular interesting case is when

$$a_2 = a_2' = 1, \tag{18}$$

which gives, because of Eq.(14) with a choice of signs,

$$a_3 = -a_6, \; a_4 = a_6, \; a_3' = -a_6', \; a_4' = a_6'. \tag{19}$$

These lead to

$$sin\theta_c = \frac{\left(\frac{m_d}{m_s}\right)^{1/2} - \left(\frac{m_u}{m_c}\right)^{1/2}}{\left(1+\frac{m_d}{m_s}\right)^{1/2}\left(1+\frac{m_u}{m_c}\right)^{1/2}}. \tag{20}$$

As well-known, this is quite close to the measured value for the Cabibbo angle.[5] The mixing angle θ_c is a dynamical signature in the group space, pointing to that direction which seeks out the residual S_2 symmetry. Although at this time we have not been able to associate any deeper meaning to this choice, other than the fact that the values for $a_{2,3,4}$ look quite symmetrical, it does illustrate succinctly the capability to relate to data.

We may wonder whether there is any freedom in introducing further terms for the light sector. In other words, is there a δM, which is simultaneously diagonalizable with M_1 in the sense of Eq.(6)? By using $g^2 = e$, one can show that the only necessary condition is

$$\delta M M_1^\dagger M_1 = M_1 M_1^\dagger \delta M, \tag{21}$$

which can be solved to give

$$\delta M = h_1 C_2 + h_2((123) - (132)), \tag{22}$$

where $h_{1,2}$ are some arbitrary constants. This matrix is also simultaneously diagonalised with M_0 and therefore does not lead to any CKM heavy light mixing. Besides, there is no underlying group argument as we had for M_1 to justify its being. We shall just discard it.

To discuss the CKM heavy light mixing, it is convenient to make a unitary transformation to decompose into the irreducible subspaces, viz. $3 \to 1 \oplus 2$. This is done by

$$g_i \to \mathcal{U}^\dagger g_i \mathcal{U},$$

where

$$\mathcal{U} = \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}. \tag{23}$$

Then, the mass matrix

$$M_0 + M_1 \to \begin{pmatrix} (M_1)_{2\times 2} & 0_{2\times 1} \\ 0_{1\times 2} & \bar{m}_0 \end{pmatrix},$$

in which $\bar{m}_0 = 3m_0 + m_1$ and

$$(M_1)_{2\times 2} = m_1 \left(\frac{\sqrt{3}}{2}(a_2 - a_3)\sigma_1 + \sqrt{3}a_6 i\sigma_2 + \frac{1}{2}(-a_2 - a_3 + 2a_4)\sigma_3 \right). \tag{24}$$

We make the ansatz that heavy light transition is due to

$$M_2 = \begin{pmatrix} 0 & 0 & \Delta f_x \\ 0 & 0 & \Delta f_y \\ \Delta d_x & \Delta d_y & 0 \end{pmatrix}, \tag{25}$$

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in which Δd's and Δf's are complex numbers of order at most m_1, so that all low energy flavor changing neutral processes due to the absorption, emission or exchange of attendant Higgs scalars will be suppressed by heavy quark propagators.

We are now ready to complete our discussion of the CKM matrix by performing an expansion in inverse powers of m_b and m_t.[6] We note that for $M_u = M_0 + \epsilon M_1 + \epsilon M_2$. we have

$$M_u M_u^\dagger = \bar{m}_0^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon \bar{m}_0 \begin{pmatrix} 0 & 0 & \Delta f_x \\ 0 & 0 & \Delta f_y \\ \Delta f_x^\star & \Delta f_y^\star & 0 \end{pmatrix} + O(\epsilon^2). \tag{26}$$

ϵ is a counting parameter in the inverse mass expansion, which will be set to unity afterwards. Note that because we are dealing with left-left mixing, the second term on the right hand side of the last equation, which is the only $O(\epsilon)$ term, has dependence on Δf's only. Δd's are not measurable to this order.

It is a simple matter to solve for the eigenvectors to obtain

$$|u, c> = |\lambda_{1,2}> \cong |\lambda_{1,2}^0> - \frac{\Delta F_{1,2}^\star}{m_t}|\lambda_3^0>,$$

$$|t> = |\lambda_3> \cong |\lambda_3^0> + \frac{\Delta F_1}{m_t}|\lambda_1^0> + \frac{\Delta F_2}{m_t}|\lambda_2^0>,$$

where

$$\Delta F_{1,2} \equiv <\lambda_{1,2}^0|\mathcal{U} \begin{pmatrix} \Delta f_x \\ \Delta f_y \\ 0 \end{pmatrix} = -N_{1,2}(\sqrt{\frac{3}{2}}(x+y)_{1,2}\Delta f_x + \sqrt{\frac{1}{2}}(x-y)_{1,2}\Delta f_y). \tag{27}$$

From these, we form the CKM matrix elements

$$V_{ud} = <u|d> \cong <\lambda_1^0|\lambda_2'^0> = cos\theta_c,$$

$$V_{us} \cong sin\theta_c, \quad V_{cd} \cong -sin\theta_c, \quad V_{cs} \cong cos\theta_c,$$

$$V_{td} \cong \frac{\Delta F_1^\star}{m_t}cos\theta_c - \frac{\Delta F_2^\star}{m_t}sin\theta_c - \frac{\Delta F_1'^\star}{m_b},$$

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$$V_{ts} \cong \frac{\Delta F_1^\star}{m_t} sin\theta_c + \frac{\Delta F_2^\star}{m_t} cos\theta_c - \frac{\Delta F_2^{'\star}}{m_b},$$

$$V_{ub} \cong -V_{td}^\star cos\theta_c - V_{ts}^\star sin\theta_c,$$

$$V_{cb} \cong V_{td}^\star sin\theta_c - V_{ts}^\star cos\theta_c,$$

$$V_{tb} \cong 1. \tag{28}$$

These expressions have further corrections of order $\frac{1}{m_b^2}$, $\frac{1}{m_b m_t}$, $\frac{1}{m_t^2}$. Eqs.(28) may be taken as a slightly generalized Wolfenstein parameterization.[7] If we assume $\Delta F_{1,2}/m_t \ll \Delta F_{1,2}'/m_b$ and drop the former, the number of parameters we need to incorporate heavy-light transitions in CKM matrix is three, namely the magnitudes of $\Delta f_{x,y}'$ and the relative phase, which is precisely what we need to specify in general. CP violation is intimately tied up with flavor violation in the heavy-light connection.

Because of simultaneous diagonalizability of M_0 and M_1, there is no flavor changing neutral current due to tree level scalar exchanges in the light sector. The masses of those scalar doublets associated with M_0 and M_1 can take on single Higgs values $\sim m_W$ as in conventional Standard Model analysis. Particularly, they will not give rise to disproportionate surprises in K^0-$\bar{K}^0$ or D^0-$\bar{D}^0$ systems.[2] New physics most likely will be first revealed in processes through the intermediary of top and bottom quarks, whence exploration in future B-factories should be most interesting. We are looking into phenomenological manifestation of the terms Δd, $\Delta d'$, Δf, $\Delta f'$ and the accompanying scalars.

In summary, we have argued that if the flavor space admits an approximate symmetry of a finite group, then the dominant piece of the Yukawa interactions should be a function of some class operators of that group. Ratios of light quark masses and the Cabbibo angle are given by directional parameters of some subgroup into which the original symmetry breaks. The dynamical issue of masses and mixing is then shifted into the eventual determination of these parameters from some first principle. S_3 is used to show explicitly how this works.

We have been able to match the independent parameters in the analysis to basically quark masses and CKM angles. There is no flavor changing neutral current, until the last stage when heavy-light transition terms are introduced to account for heavy-light CKM mixing.

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